Bellcomm

955 L'Enfant Plaza North, S.W.
Washington, D. C. 20024
B72 01013

date: January 31, 1972

to: Distribution

from: S. Y. Lee

subject: Optimum Post-Filters and Minimum Mean Square Errors of Line Scanning Systems
Case 620

ABSTRACT

By the use of basic techniques which have been developed for treating linear filters in network theory, optimum continuous one-dimensional and semi-discrete two-dimensional post-filters for a line scanning image system are derived based on the mean square error criterion. Explicit formulas in terms of the scanning system parameters have been developed for the calculations of the minimum mean square errors when either continuous one-dimensional or semi-discrete two-dimensional post-filtering is performed. Utilizing these results, the relative advantage of the two-dimensional semi-discrete post-filtering over the continuous one-dimensional post-filtering is obtained. This is shown by an illustrative example.

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MEMORANDUM FOR FILE

I. INTRODUCTION

In Reference [1], mathematical relationships of scanning are derived for the two-dimensional semi-discrete models. In this memorandum we utilize these results as a tool for the determination of the optimum post-filters shown in Figure 1 when the mean square error criterion is adopted. Since the analysis of the line scanning image system in the spatial frequency domain is much simpler than the analysis in the spatial domain, optimization will be performed in terms of spatial transfer functions.

II. OPTIMIZATION OF TWO-DIMENSIONAL SEMI-DISCRETE POST-FILTERS

The model given in Figure 1 enables one to evaluate system performance and to employ optimum post-filtering techniques as stated in Reference [1]. To derive the optimum two-dimensional post-filters in the mean square sense, first express the error in terms of semi-discrete signals,

$$\tilde{\varepsilon} = \tilde{f}_1 - \tilde{f}_4 = \tilde{f}_1 - (\tilde{f}_3+\tilde{n}) * \tilde{a}_4 \qquad (1)$$

where the variables are defined in Figure 1.

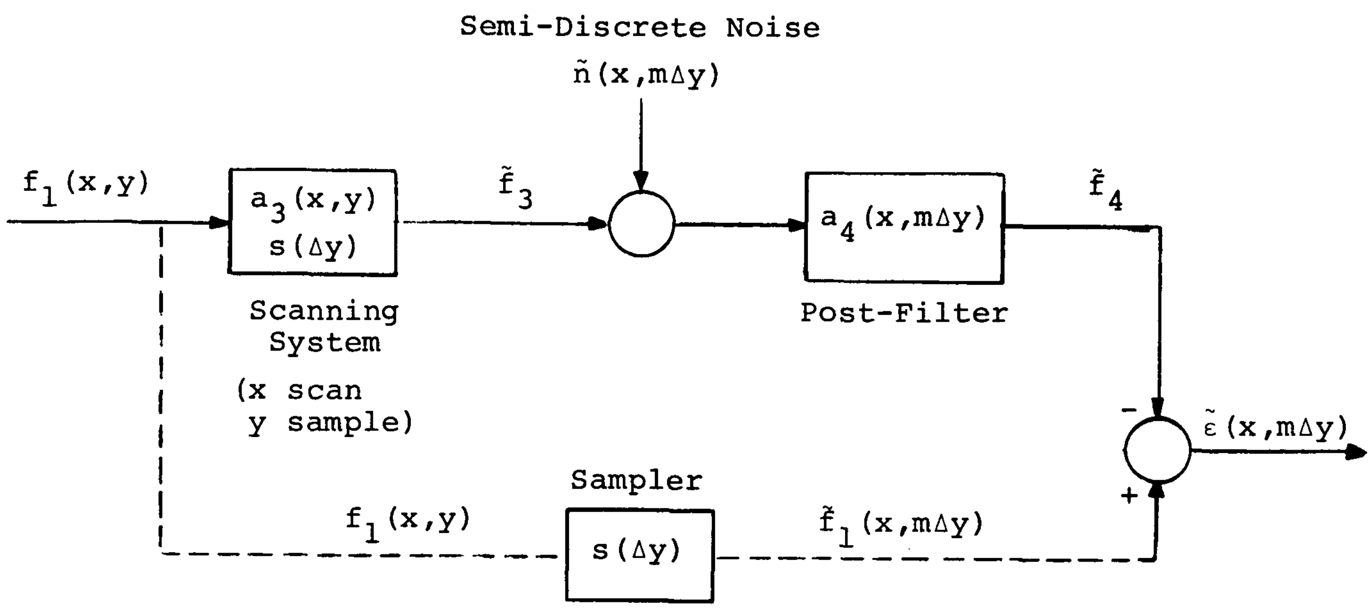

FIGURE 1 - MODEL FOR PERFORMANCE EVALUATION AND POST-FILTERING

The ensemble autocorrelation function for a semi-discrete stationary signal is defined as

$$\tilde{A}_f(x,m\Delta y) \quad = \quad E[\tilde{f}(x+a,m\Delta y+n\Delta y)\tilde{f}(a,n\Delta y)] \quad = \quad E[f_+ f] \ . \qquad (2)$$

From (2) and (1), the autocorrelation function of the error signal is

$$\tilde{A}_\varepsilon \quad = \quad E[\tilde{\varepsilon}_+ \tilde{\varepsilon}] \qquad (3)$$

$$= \quad E\{[\tilde{f}_{1+} - (\tilde{f}_3 \tilde{*} \tilde{a}_4)_+ - (\tilde{n} \tilde{*} \tilde{a}_4)_+][\tilde{f}_1 - (\tilde{f}_3 \tilde{*} \tilde{a}_4) - (\tilde{n} \tilde{*} \tilde{a}_4)]\} \ .$$

Expanding (3) and assuming the signal and noise are uncorrelated and have zero mean, that is

$$E[\tilde{n}_{+}\tilde{f}_{1}] = E[\tilde{n}\tilde{f}_{1+}] = E[\tilde{n}] = E[\tilde{f}] = 0 \tag{4}$$

we obtain

$$\tilde{A}_{\varepsilon} = E[(\tilde{f}_3\tilde{*}\tilde{a}_4)_{+}(\tilde{f}_3\tilde{*}\tilde{a}_4)] + E[(\tilde{n}\tilde{*}\tilde{a}_4)_{+}(\tilde{n}\tilde{*}\tilde{a}_4)] - E[\tilde{f}_{1+}(\tilde{f}_3\tilde{*}\tilde{a}_4)] \\ - E[\tilde{f}_1(\tilde{f}_3\tilde{*}\tilde{a}_4)_{+}] + E[\tilde{f}_{1+}\tilde{f}_1] \ . \tag{5}$$

Applying (2) and noting that (see Reference [1])

$$\begin{aligned} A_{f_3} &= A_{f_1} * a_3 * a_{3-} \\ \tilde{A}_{f_3} &= \tilde{A}_{f_1} \tilde{*} \tilde{a}_3 \tilde{*} \tilde{a}_{3-} \\ C_{f_{1+}f_3} &= A_{f_1} * a_{3-} \\ C_{f_1 f_{3+}} &= A_{f_1} * a_3 \\ \tilde{C}_{f_{1+}f_3} &= \tilde{A}_{f_1} \tilde{*} \tilde{a}_{3-} \\ \tilde{C}_{f_1 f_{3+}} &= \tilde{A}_{f_1} \tilde{*} \tilde{a}_3 \end{aligned} \tag{6}$$

(5) becomes

$$\tilde{A}_{\varepsilon} = (\tilde{A}_{f_3} + \tilde{A}_n)\tilde{*}\tilde{a}_4\tilde{*}\tilde{a}_{4-} - \tilde{C}_{f_{1+}f_3}\tilde{*}\tilde{a}_{4-} - \tilde{C}_{f_1 f_{3+}}\tilde{*}\tilde{a}_4 + \tilde{A}_{f_1} \tag{7}$$

or in the spatial frequency domain, we again apply the results of Reference [1]

$$
\begin{aligned}
S_{f_3} &= S_{f_1}|A_3|^2 \\
\tilde{S}_{f_3} &= \tilde{S}_{f_1}|\tilde{A}_3|^2 \\
S_{f_{1+}f_3} &= S_{f_1}\bar{A}_3 \\
S_{f_1f_{3+}} &= S_{f_1}A_3 \\
\tilde{S}_{f_{1+}f_3} &= \tilde{S}_{f_1}\bar{\tilde{A}}_3 \\
\tilde{S}_{f_1f_{3+}} &= \tilde{S}_{f_1}\tilde{A}_3
\end{aligned}
\tag{8}
$$

to (7) and obtain

$$\tilde{S}_\varepsilon = (\tilde{S}_{f_3}+\tilde{S}_n)|\tilde{A}_4|^2 - \tilde{S}_{f_{1+}f_3}\bar{\tilde{A}}_4 - \tilde{S}_{f_1f_{3+}}\tilde{A}_4 + \tilde{S}_{f_1} \tag{9}$$

Equation (9) can be rewritten as

$$\tilde{S}_\varepsilon = \left| (\tilde{S}_{f_3}+\tilde{S}_n)^{1/2}\tilde{A}_4 - \frac{\tilde{S}_{f_{1+}f_3}}{(\tilde{S}_{f_3}+\tilde{S}_n)^{1/2}} \right|^2 - \frac{|\tilde{S}_{f_1f_{3+}}|^2}{(\tilde{S}_{f_3}+\tilde{S}_n)} + \tilde{S}_{f_1} \; . \tag{10}$$

It is evident from Figure 1 that the variation in $\tilde{A}_4$ does not change the functions of $\tilde{S}_{f_3}$, $\tilde{S}_n$, $\tilde{S}_{f_{1+}f_3}$ and $\tilde{S}_{f_1}$. Thus, the optimum semi-discrete two-dimensional post-filter transfer function $\tilde{A}_{4(opt)}$ can be readily obtained from (10) as

$$\tilde{A}_{4(opt)} = \frac{\tilde{S}_{f_{1+}f_3}}{\tilde{S}_{f_3}+\tilde{S}_n} \tag{11}$$

The mean square value of a semi-discrete signal is derived in Reference [1] as

$$E[\tilde{f}^2] = \tilde{A}_f(0,0) = \int_{-\infty}^{\infty} \int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \tilde{S}_f(w_x,w_y)\,dw_y dw_x \tag{12}$$

where Δy is the interval between samples in the y coordinate. Hence, from (12) and (10), the minimum mean square error after optimum post-filtering is

$$E[\tilde{\varepsilon}^2]_{min} = \int_{-\infty}^{\infty} \int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \tilde{S}_{f_1}(w_x,w_y)\,dw_y dw_x - \int_{-\infty}^{\infty} \int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \frac{|\tilde{S}_{f_1 f_{3+}}(w_x,w_y)|^2}{\tilde{S}_{f_3}(w_x,w_y)+\tilde{S}_n(w_x,w_y)}\,dw_y dw_x \;. \tag{13}$$

From (8), the optimum semi-discrete two-dimensional post-filter transfer function $\tilde{A}_{4(opt)}$ of (11) and the minimum mean square error after optimum post-filtering $E[\tilde{\varepsilon}^2]_{min}$ of (13) can be rewritten in terms of the parameters of the scanning system respectively as

$$\tilde{A}_4(w_x,w_y)_{opt} = \frac{\bar{\tilde{A}}_3(w_x,w_y)\tilde{S}_{f_1}(w_x,w_y)}{|\tilde{A}_3(w_x,w_y)|^2\tilde{S}_{f_1}(w_x,w_y) + \tilde{S}_n(w_x,w_y)} \tag{14}$$

and

$$E[\tilde{\varepsilon}^2]_{min} = \int_{-\infty}^{\infty} \int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \tilde{S}_{f_1}(w_x,w_y)\,dw_y dw_x - \int_{-\infty}^{\infty} \frac{|\tilde{A}_3(w_x,w_y)\tilde{S}_{f_1}(w_x,w_y)|^2}{\tilde{S}_{f_1}(w_x,w_y)\,|\tilde{A}_3(w_x,w_y)|^2 + \tilde{S}_n(w_x,w_y)}\,dw_y dw_x \;. \tag{15}$$

III. OPTIMIZATION OF ONE-DIMENSIONAL POST-FILTER

In many practical situations, the post-filter may be only realizable as a one-dimensional filter. For these cases, it is necessary to derive the optimum one-dimensional post-filter and the corresponding minimum mean square error when the input signal is two-dimensional. It should be noted that (10) holds for either one or two-dimensional post-filters. However, it is evident that no one-dimensional filter of the form of

$$\tilde{a}_4(x, m\Delta y) \equiv \begin{cases} a_4'(x) & \text{if } m=0 \\ \\ 0 & \text{if } m \neq 0 \end{cases}, \qquad (16)$$

where $a_4'(x)$ is the continuous one-dimensional impulse response of the post-filter, can make the first term of (10) vanish since the other parameters involved are inherently two-dimensional. Thus, the minimum mean square error after optimum one-dimensional post-filtering will be greater than the minimum mean square error if optimum two-dimensional post-filtering is performed. This will be illustrated below and in Section IV.

By noting that the procedure of obtaining the optimum post-filter involves differentiation of the mean square error of (10) with respect to the post-filter $\tilde{A}_4$, the last two terms on the right-hand side of (10) vanish and can be ignored. Thus, to minimize the mean square error with one-dimensional post-filtering, we rewrite the first term on the right-hand side of (10) as

$$\tilde{S}_{1\varepsilon}(w_x,w_y) = \left| (\tilde{S}_{f_3}+\tilde{S}_n)^{1/2}\tilde{A}_4 - \frac{\tilde{S}_{f_{1+}f_3}}{(\tilde{S}_{f_3}+\tilde{S}_n)} \right|^2$$

$$= (\tilde{S}_{f_3}+\tilde{S}_n)|\tilde{A}_4|^2 - \tilde{S}_{f_1f_{3+}}\tilde{A}_4 - \tilde{S}_{f_{1+}f_3}\bar{\tilde{A}}_4 + \frac{|\tilde{S}_{f_{1+}f_3}|^2}{(\tilde{S}_{f_3}+\tilde{S}_n)} \quad . \tag{17}$$

It should be noted that (17) is in terms of semi-discrete spectra. To write (17) in terms of one-dimensional spectra, we use the relationships between $\tilde{A}_4(w_x,w_y)$ and $A_4'(w_x)$, and between the semi-discrete and one-dimensional spectra, which are derived in the Appendix as

$$\tilde{A}_4(w_x,w_y) = \Delta y\ A_4'(w_x) \tag{18}$$

and

$$S_f'(w_x) = \int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \tilde{S}_f(w_x,w_y)\,dw_y \tag{19}$$

respectively. Thus, the one-dimensional equivalent of (17) is

$$S_{1\varepsilon}'(w_x) = (\Delta y)^2|A_4'|^2 \int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} (\tilde{S}_{f_3}+\tilde{S}_n)\,dw_y - \Delta y A_4' \int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \tilde{S}_{f_1f_{3+}}\,dw_y$$

$$- \Delta y \bar{A}_4' \int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \tilde{S}_{f_{1+}f_3}\,dw_y + \int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \frac{|\tilde{S}_{f_{1+}f_3}|^2}{(\tilde{S}_{f_3}+\tilde{S}_n)}\,dw_y \tag{20}$$

or rearranging we have

$$S'_{1\varepsilon}(w_x) = \left| \left[(\Delta y)^2 \int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} (\tilde{S}_{f_3}+\tilde{S}_n)\,dw_y \right]^{1/2} A'_4 - \frac{\Delta y \int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \tilde{S}_{f_{1+}f_3}\,dw_y}{\left[(\Delta y)^2 \int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} (\tilde{S}_{f_3}+\tilde{S}_n)\,dw_y \right]^{1/2}} \right|^2$$

$$+ \int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \frac{\left|\tilde{S}_{f_{1+}f_3}\right|}{(\tilde{S}_{f_3}+\tilde{S}_n)}\,dw_y - \frac{\left| \int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \tilde{S}_{f_{1+}f_3}\,dw_y \right|^2}{\int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} (\tilde{S}_{f_3}+\tilde{S}_n)\,dw_y} \,. \tag{21}$$

The optimum one-dimensional post-filter can be obtained by differentiating (21) with respect to A'_4 and setting the resulting equation to zero,

$$A'_4(w_x)_{opt} = \frac{\int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \tilde{S}_{f_{1+}f_3}(w_x,w_y)\,dw_y}{\Delta y \int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} [\tilde{S}_{f_3}(w_x,w_y) + \tilde{S}_n(w_x,w_y)]\,dw_y} \tag{22}$$

From (8), (22) can be rewritten in terms of the parameters of the scanning system as

$$A'_4(w_x)_{opt} = \frac{\int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \tilde{\bar{A}}_3(w_x,w_y)\tilde{S}_{f_1}(w_x,w_y)\,dw_y}{\Delta y \int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} [\tilde{S}_{f_1}(w_x,w_y)|\tilde{A}_3(w_x,w_y)|^2 + \tilde{S}_n(w_x,w_y)]\,dw_y} \,. \tag{23}$$

The mean square value of a one-dimensional signal is defined as

$$E[(f')^2] = A_f'(0) = \int_{-\infty}^{\infty} S_f'(w_x)dw_x \ . \tag{24}$$

Hence, from (10), (21) and (22), the minimum mean square error with one-dimensional post-filtering is obtained

$$E[(\varepsilon')^2]_{min} = \int_{-\infty}^{\infty}\int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \tilde{S}_{f_1} dw_y dw_x - \int_{-\infty}^{\infty}\int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \frac{|\tilde{S}_{f_{1+}f_3}|^2}{\tilde{S}_{f_3}+\tilde{S}_n} dw_y dw_x$$

$$+ \int_{-\infty}^{\infty}\int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \frac{|\tilde{S}_{f_{1+}f_3}|^2}{(\tilde{S}_{f_3}+\tilde{S}_n)} dw_y dw_x - \int_{-\infty}^{\infty} \frac{\left|\int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \tilde{S}_{f_{1+}f_3} dw_y\right|^2}{\int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} (\tilde{S}_{f_3}+\tilde{S}_n) dw_y} dw_x \tag{25}$$

or

$$E[(\varepsilon')^2]_{min} = \int_{-\infty}^{\infty}\int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \tilde{S}_{f_1} dw_y dw_x - \int_{-\infty}^{\infty} \frac{\left|\int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \tilde{S}_{f_{1+}f_3} dw_y\right|^2}{\int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} (\tilde{S}_{f_3}+\tilde{S}_n) dw_y} dw_x \ . \tag{26}$$

From (8), (26) can be expressed in terms of the parameters of the scanning system as

$$E[(\varepsilon')^2]_{min} = \int_{-\infty}^{\infty} \int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \tilde{S}_{f_1}(w_x,w_y)\,dw_y\,dw_x$$

$$- \int_{-\infty}^{\infty} \frac{\left| \int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \tilde{A}_3(w_x,w_y)\,\tilde{S}_{f_1}(w_x,w_y)\,dw_y \right|^2}{\int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} [\tilde{S}_{f_1}(w_x,w_y)\,|\tilde{A}_3(w_x,w_y)|^2 + \tilde{S}_n(w_x,w_y)]\,dw_y}\,dw_x \; . \tag{27}$$

The advantage of two-dimensional semi-discrete processing over one-dimensional continuous processing can be determined by subtracting (15) from (27),

$$E[(\varepsilon')^2]_{min} - E[(\tilde{\varepsilon})^2]_{min}$$

$$= \int_{-\infty}^{\infty} \frac{\left| \int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \tilde{A}_3(w_x,w_y)\,\tilde{S}_{f_1}(w_x,w_y)\,dw_y \right|^2}{\int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} [\tilde{S}_{f_1}(w_x,w_y)\,|\tilde{A}_3(w_x,w_y)|^2 + \tilde{S}_n(w_x,w_y)]\,dw_y}\,dw_x \tag{28}$$

$$- \int_{-\infty}^{\infty} \int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \frac{|\tilde{A}_3(w_x,w_y)\,\tilde{S}_{f_1}(w_x,w_y)|^2}{\tilde{S}_f(w_x,w_y)\,|\tilde{A}_3(w_x,w_y)|^2 + \tilde{S}_n(w_x,w_y)}\,dw_y\,dw_x \; .$$

IV. EXAMPLE

The model given in Figure 1 enables one to evaluate system performance after post-filtering. To compare the relative advantage of two-dimensional semi-discrete post-filtering over one-dimensional continuous post-filtering, we should first find the optimum one-dimensional and two-dimensional semi-discrete post-filters for the model given in Figure 2. Then, to simplify calculations, the minimum mean square errors for both techniques are determined and compared for the noiseless system.

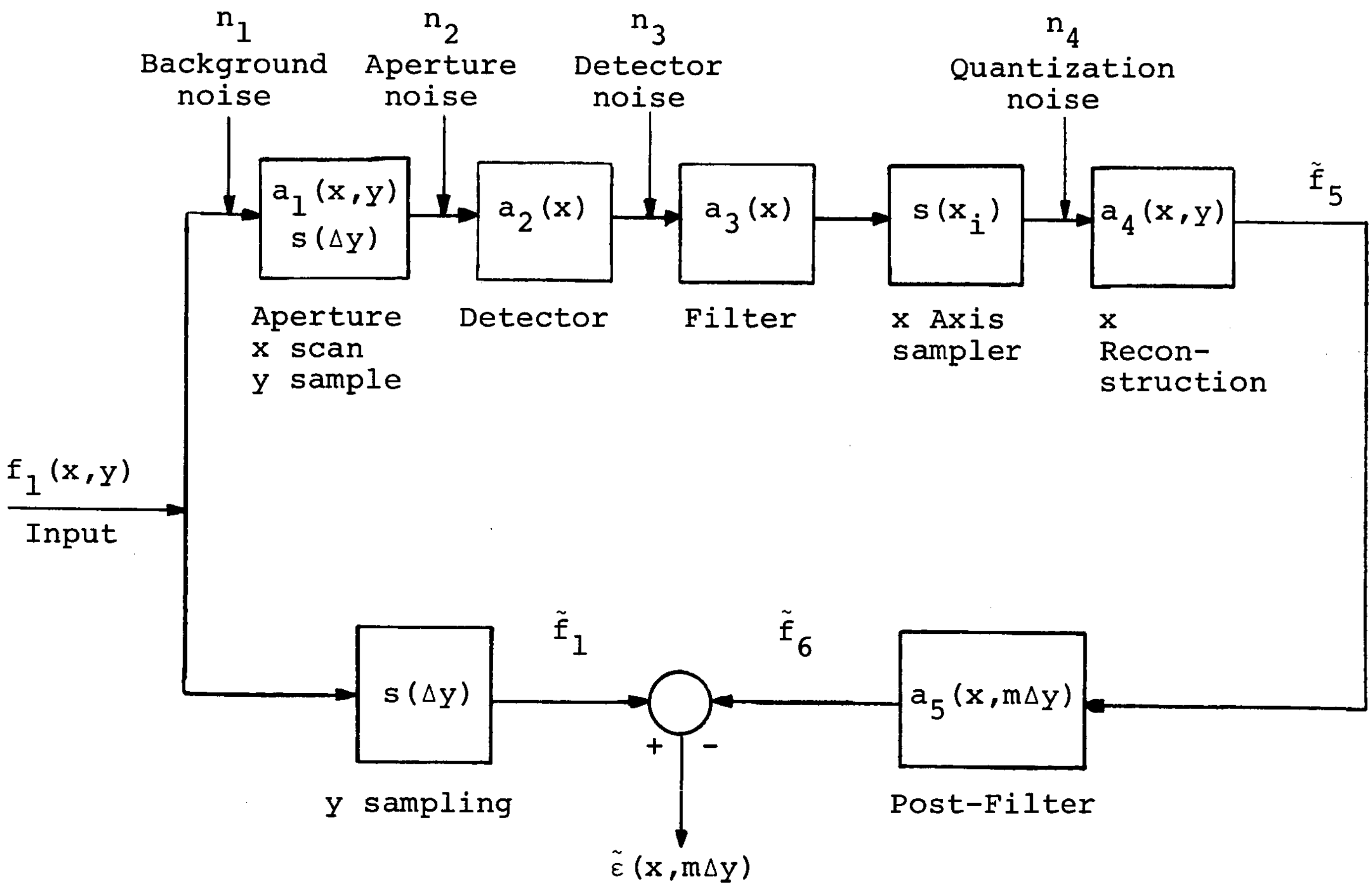

FIGURE 2 - MODEL OF A LINE SCANNING SYSTEM

The correlation spectra of the output signal of the reconstruction filter $a_4(x,y)$, shown in Figure 2, have been determined in Reference [1] as

$$\tilde{S}_{f_5} = (\tilde{S}_{f_1}+\tilde{S}_{n_1})|\tilde{A}_1\tilde{A}_2\tilde{A}_3\tilde{A}_4|^2 + \tilde{S}_{n_2}|\tilde{A}_2\tilde{A}_3\tilde{A}_4|^2 + \tilde{S}_{n_3}|\tilde{A}_3\tilde{A}_4|^2 + \tilde{S}_{n_4}|\tilde{A}_4|^2 \tag{29}$$

$$\tilde{S}_{f_1 f_{5+}} = \tilde{S}_{f_1}\tilde{A}_1\tilde{A}_2\tilde{A}_3\tilde{A}_4 \tag{30}$$

and

$$\tilde{S}_{f_{1+} f_5} = \tilde{S}_{f_1}\bar{\tilde{A}}_1\bar{\tilde{A}}_2\bar{\tilde{A}}_3\bar{\tilde{A}}_4 \ . \tag{31}$$

Associating $\tilde{f}_5$ and a_5 of Figure 2 with signal $\tilde{f}_3$ and a_4 of Figure 1, respectively, we obtain the optimum two-dimensional semi-discrete post-filter by substituting (29) and (31) into (11)

$$\tilde{A}_{5(\mathrm{opt})} = \frac{\tilde{S}_{f_1}\bar{\tilde{A}}_1\bar{\tilde{A}}_2\bar{\tilde{A}}_3\bar{\tilde{A}}_4}{\tilde{S}_{f_5}} \ .$$

The corresponding minimum mean square error is obtained from (13), (29) and (30) as

$$E[(\tilde{\varepsilon})^2]_{\min} = \int_{-\infty}^{\infty}\int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \left[\tilde{S}_{f_1}(w_x,w_y) - \frac{|\tilde{S}_{f_1 f_{5+}}(w_x,w_y)|^2}{\tilde{S}_{f_5}}\right] dw_y dw_x \ . \tag{33}$$

Assuming the model of Figure 2 is noiseless, then from (29), the optimum two-dimensional semi-discrete post-filter reduces to

$$\tilde{A}_5(w_x,w_y)_{opt} = \frac{\tilde{S}_{f_1}\bar{\tilde{A}}_1\bar{\tilde{A}}_2\bar{\tilde{A}}_3\bar{\tilde{A}}_4}{\tilde{S}_{f_1}|\tilde{A}_1\tilde{A}_2\tilde{A}_3\tilde{A}_4|^2} = \frac{1}{\tilde{A}_1\tilde{A}_2\tilde{A}_3\tilde{A}_4} \tag{34}$$

and the corresponding minimum mean square error after optimum post-filter for the noiseless system is

$$E[(\tilde{\varepsilon})^2]_{min} = \int_{-\infty}^{\infty}\int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}}\left[\tilde{S}_{f_1} - \frac{\tilde{S}_{f_1}^{2}|\tilde{A}_1\tilde{A}_2\tilde{A}_3\tilde{A}_4|^2}{\tilde{S}_{f_1}|\tilde{A}_1\tilde{A}_2\tilde{A}_3\tilde{A}_4|^2}\right]dw_y dw_x = 0\ . \tag{35}$$

The optimum one-dimensional post-filter can be derived by associating Figures 1 and 2 and by substituting (29) and (31) into (23)

$$A_5'(w_x)_{opt} = \frac{\int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}}\tilde{S}_{f_1+f_5}(w_x,w_y)dw_y}{\Delta y\int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}}\tilde{S}_{f_5}(w_x,w_y)dw_y} \tag{36}$$

and the corresponding minimum mean square error is obtained from (26), (29) and (31) as

$$E[(\varepsilon')^2]_{min} = \int_{-\infty}^{\infty}\int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}}\tilde{S}_{f_1}(w_x,w_y)dw_y dw_x$$
$$- \int_{-\infty}^{\infty}\frac{\left|\int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}}\tilde{S}_{f_1+f_5}(w_x,w_y)dw_y\right|^2}{\int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}}\tilde{S}_{f_5}(w_x,w_y)dw_y}dw_x\ . \tag{37}$$

Assuming the model of Figure 2 again noiseless, then from (29), the optimum one-dimensional post-filter reduces to

$$A_5'(w_x)_{opt} = \frac{\displaystyle\int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \tilde{S}_{f_1} \bar{\tilde{A}}_1 \bar{\tilde{A}}_2 \bar{\tilde{A}}_3 \bar{\tilde{A}}_4 \, dw_y}{\displaystyle\Delta y \int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \tilde{S}_{f_1} |\tilde{A}_1 \tilde{A}_2 \tilde{A}_3 \tilde{A}_4|^2 \, dw_y} \tag{38}$$

and the corresponding minimum mean square error after optimum post-filter for the noiseless system is

$$E[(\varepsilon')^2]_{min} = \int_{-\infty}^{\infty} \int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \tilde{S}_{f_1}(w_x, w_y) \, dw_y dw_x - \int_{-\infty}^{\infty} \frac{\left| \displaystyle\int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \tilde{S}_{f_1} \bar{\tilde{A}}_1 \bar{\tilde{A}}_2 \bar{\tilde{A}}_3 \bar{\tilde{A}}_4 \, dw_y \right|^2}{\displaystyle\int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \tilde{S}_{f_1} |\tilde{A}_1 \tilde{A}_2 \tilde{A}_3 \tilde{A}_4|^2 \, dw_y} \, dw_x \, . \tag{39}$$

Thus, by comparing the minimum mean square errors of (35) and (39), it is seen that the semi-discrete post-filtering technique has the advantage over the one-dimensional continuous post-filtering technique.

S. Y. Lee

1031-SYL-sje S. Y. Lee

Attachment
Appendix

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APPENDIX

THE RELATIONSHIPS BETWEEN THE SEMI-DISCRETE AND ONE-DIMENSIONAL PROCESS

A one-dimensional filter can be defined in terms of the semi-discrete two-dimensional filter as follows:

$$\tilde{a}(x,m\Delta y) \equiv \begin{cases} a'(x) & \text{if } m=0 \\ 0 & \text{if } m\neq 0 \end{cases} \tag{A-1}$$

where $a'(x)$ is the continuous one-dimensional impulse system response function with the Fourier spectrum $A'(w_x)$ defined as

$$A'(w_x) = \int_{-\infty}^{\infty} a'(x)e^{-j2\pi w_x x}dx \tag{A-2}$$

and $\tilde{a}(x,m\Delta y)$ is the semi-discrete two-dimensional impulse system response function whose Fourier spectrum is defined in Reference [1] as

$$\tilde{A}(w_x,w_y) = \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \tilde{a}(x,m\Delta y)e^{-j2\pi(w_x x+w_y m\Delta y)}dx\Delta y \tag{A-3}$$

Hence, the relationship between the semi-discrete spectrum $\tilde{A}(w_x,w_y)$ and the continuous one-dimensional spectrum $A'(w_x)$ is easily obtained from (A-2) and (A-3),

$$\tilde{A}(w_x,w_y) = \Delta y \; A'(w_x) \; . \tag{A-4}$$

The spatial autocorrelation function of a one-dimensional signal is defined as

$$A_f'(x) = E[f'(x+a)f'(a)] \tag{A-5}$$

and its Fourier transform pair is

$$\begin{aligned} S_f'(w_x) &= \int_{-\infty}^{\infty} A_f'(x)e^{-j2\pi w_x x}dx \\ A_f'(x) &= \int_{-\infty}^{\infty} S_f'(w_x)e^{j2\pi w_x x}dw_x \; . \end{aligned} \tag{A-6}$$

The spatial autocorrelation function of a semi-discrete two-dimensional signal is defined in Reference [1] as

$$\tilde{A}_f(x,m\Delta y) = E[\tilde{f}(x+a,m\Delta y+n\Delta y)\tilde{f}(a,n\Delta y)] \tag{A-7}$$

and its Fourier transform pair is

$$\begin{aligned} \tilde{S}_f(w_x,w_y) &= \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \tilde{A}_f(x,m\Delta y)\exp[-2\pi j(w_x x+w_y m\Delta y)]dx\Delta y \\ \tilde{A}_f(x,m\Delta y) &= \int_{-\infty}^{\infty} \int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \tilde{S}_f(w_x,w_y)\exp[2\pi j(w_x x+w_y m\Delta y)]dw_y dw_x \; . \end{aligned} \tag{A-8}$$

Thus, the relationship between the semi-discrete and the one-dimensional continuous correlation function is obtained directly from (A-5) and (A-7) as

$$\tilde{A}_f(x,0) = A'_f(x) \; . \tag{A-9}$$

Furthermore, it should be noted that from (A-8)

$$\tilde{A}_f(x,0) = \int_{-\infty}^{\infty} \left[\int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \tilde{S}_f(w_x, w_y)\, dw_y \right] \exp[2\pi j w_x x]\, dw_x \; . \tag{A-10}$$

Therefore, by comparing (A-10) with (A-6), the relationship between the one-dimensional and semi-discrete two-dimensional spectra is determined as

$$S'_f(w_x) = \int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \tilde{S}_f(w_x, w_y)\, dw_y \; .$$

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